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Harmonic oscillations of a rigid punch on a porous elastic layer☆

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ABSTRACT

The plane dynamic contact problem of the harmonic oscillations of a rigid punch on the free surface of an elastic layer of porous isotropic material with linear properties is considered. The Fourier transformation of the problem is reduced to a Fredholm integral equation of the first kind in the contact pressure. The properties of the kernel of the fundamental integral equation are investigated and a numerical method of solving it is constructed. Numerical results are compared with existing results in classical limiting cases.

A new theory of porous linear elastic materials with voids[1,2] is based on the energy-balance equation, where the presence of pores implies an additional degree of freedom, namely the relative change in an elementary volume of the material. As a consequence, the mass density is expressed in the form of the product of two quantities – the relative porosity and the mass density of the main material. This theory describes the behaviour of some geological rocks, composite materials, ceramic materials and porous metals quite well.

A general theory of such materials has been developed fairly extensively, particularly in the static case, but only a small number of problems have been solved in the dynamic case. Nevertheless, investigations of the dynamic properties of a number of porous elastic solids under contact conditions are important in the ultrasonic testing of materials, when analysing the vibrations of engineering structures on elastic foundations, in soil mechanics, in seismology and in other areas.

In this paper we propose an effective method of solving the contact problem for a porous elastic layer in the two-dimensional formulation. We first give a brief review of the fundamental equations and we then demonstrate the use of the Fourier transformation, which enables the problem to be reduced to a certain integral equation.

A similar problem for a layer of finite thickness was investigated in Ref. 3 in the static case. Some interesting results on the dynamic behaviour of porous materials are presented in Refs 4-6.

1. Review of previous results and formulation of the problem

We will consider the two-dimensional problem of the behaviour of a rigid punch with a flat base, performing translational vertical oscillations with an angular frequency Ω on the upper face of a porous elastic strip ($0 \le y \le h, |x| < \infty$) when there is no friction in the contact area. The lower face of the layer is horizontal and rests without friction on an absolutely rigid foundation. Note that in the case of the harmonic oscillations of the punch considered here, this formulation of the problem may lead to the occurrence of negative contact stresses, which is physically impossible and indicates that the punch loses contact with the elastic foundation. However, the classical treatment implies that a constant vertical load acts on the punch (for example, the weight of a structure in problems of the dynamics of buildings or seismology), on which dynamic oscillations of lesser amplitude are superimposed. In this case the resultant contact stress remains positive. Then, in view of the linearity of the model, the problem can be split into two independent ones: static and dynamic. The static problem was investigated earlier[3] for a porous strip. This paper considers the analogous dynamic problem.

In the case of harmonic oscillations, all the physical quantities can be represented in the form of the real part of the corresponding complex-valued function: $\tilde{f}(x, y, t) = Re[f(x, y)\exp(-i\Omega t)]$, where $f(x,y)$ is the complex amplitude of the oscillations. In this approach the

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Cowin–Goodman–Nunziato theory[1,2] for the two-dimensional case is described by the following system of partial differential equations

$$\frac{\partial^2 u}{\partial x^2} + c^2\frac{\partial^2 u}{\partial y^2} + (1-c^2)\frac{\partial^2 \upsilon}{\partial x \partial y} + H\frac{\partial \phi}{\partial x} + k_p^2 u = 0$$
$$\frac{\partial^2 \upsilon}{\partial y^2} + c^2\frac{\partial^2 \upsilon}{\partial x^2} + (1-c^2)\frac{\partial^2 u}{\partial x \partial y} + H\frac{\partial \phi}{\partial y} + k_p^2 \upsilon = 0$$
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \left(\frac{i\omega}{\alpha}\Omega + \frac{\rho k}{\alpha}\Omega^2 - \frac{\xi}{\alpha}\right)\phi - \frac{\beta}{\alpha}\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) = 0 \tag{1.1}$$

where $u(x,y)$ and $\upsilon(x,y)$ are the components of the displacement vector and $\phi = \phi(x,y)$ is a function defining the change in the relative volume of the material compared with the initial value $\phi = \nu - \nu_0$ and is related to the porosity.[1,2] We will use the standard notation of the classical theory of elasticity: λ and μ are the moduli of elasticity, ρ is the mass density of the material and α, β, ξ, ω, k are positive physical parameters related to the porosity of the material. The remaining physical parameters in system (1.1) are defined as follows:

$$k_p = \frac{\Omega}{c_p},\ k_s = \frac{\Omega}{c_s},\ c_p^2 = \frac{\lambda + 2\mu}{\rho},\ c_s^2 = \frac{\mu}{\rho},\ H = \frac{\beta}{\lambda + 2\mu},\ c^2 = \frac{c_s^2}{c_p^2} = \frac{k_p^2}{k_s^2} < 1 \tag{1.2}$$

where c_p is the velocity of a longitudinal wave, c_s is the velocity of a transverse wave, and k_p and k_s are the corresponding wave numbers.

When $\beta = 0$ the first two equations of system (1.1) become independent of the third, which corresponds to an ordinary elastic material, irrespective of the values of the remaining constants α, ξ, ω, k.

If the functions $u(x,y)$, $\upsilon(x,y)$ and $\phi(x,y)$ are defined, the stress tensor components can be obtained from the following relations

$$\frac{\sigma_{xx}}{\lambda + 2\mu} = \frac{\partial u}{\partial x} + (1-2c^2)\frac{\partial \upsilon}{\partial y} + H\phi$$
$$\frac{\sigma_{yy}}{\lambda + 2\mu} = (1-2c^2)\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + H\phi,\quad \frac{\sigma_{xy}}{\mu} = \frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x} \tag{1.3}$$

The boundary conditions can be written in the form

$$y = 0:\ \sigma_{xy} = \upsilon = \frac{\partial \phi}{\partial y} = 0,\quad |x| < \infty$$
$$y = h:\ \sigma_{xy} = \frac{\partial \phi}{\partial y} = 0,\quad |x| < \infty;\quad \sigma_{yy} = 0,\quad |x| > a;\quad \upsilon = \upsilon_0(x),\quad |x| < a \tag{1.4}$$

Here $2a$ is the length of the punch. The boundary conditions for the porosity, i.e., the functions $\phi(x,y)$, follow from the condition of conservation of energy.[7]

2. The fundamental integral equation

Using a Fourier transformation in the variable x, the system of functional differential equations can be reduced to a system of ordinary differential equations with constant coefficients (see, for example, Ref. 8). System (1.1) and boundary conditions (1.4) in Fourier transforms, denoted by the corresponding capital letters, takes the following form

$$c^2U'' - s^2U + (1-c^2)(-is)V' - isH\Phi + k_p^2U = 0$$
$$(1-c^2)(-is)U' + V'' - c^2s^2V + H\Phi' + k_p^2V = 0$$
$$\frac{\beta}{\alpha}(isU - V') + \Phi'' + \left(\frac{i\omega}{\alpha}\Omega + \frac{\rho k}{\alpha}\Omega^2 - \frac{\xi}{\alpha} - s^2\right)\Phi = 0 \tag{2.1}$$

$$V(s,0) = \Phi'(s,0) = U'(s,0) - isV(s,0) = 0$$
$$\Phi'(s,h) = U'(s,h) - isV(s,h) = 0$$
$$(1-2c^2)(-is)U(s,h) + V'(s,h) + H\Phi = \frac{P(s)}{\lambda + 2\mu} \tag{2.2}$$

Here s is the transformation parameter, a prime denotes a derivative with respect to y, and the value of the normal stress along the upper surface of the strip $\sigma_{yy}(x,0)$ is denoted by $p(x)$. Note that $p(x) = 0$, $|x| > a$, while the unknown quantity $p(x)$, $|x| \le a$ physically denotes the contact pressure.

The characteristic equation of system (2.1)

$$\begin{vmatrix} c^2\lambda^2 + k_p^2 - s^2 & (1-c^2)(-is)\lambda & -isH \\ (1-c^2)(-is)\lambda & \lambda^2 + k_p^2 - c^2s^2 & \lambda H \\ \frac{\beta}{\alpha}is & -\frac{\beta}{\alpha}\lambda & \lambda^2 + \frac{i\omega}{\alpha}\Omega + \frac{\rho k}{\alpha}\Omega^2 - \frac{\xi}{\alpha} - s^2 \end{vmatrix} = 0$$

has six simple roots

$$\lambda = \pm\gamma_m(s), \quad \gamma_m(s) = \sqrt{s^2 - k_m^2}, \quad m = 1, 2, 3$$

$$k_{1,2}^2 = \frac{1}{2l_2^2}\left\{-[1 - N - i\omega^* k_p - (l_2^2 + k^{*2})] \pm \right.$$

$$\left. \pm\sqrt{[1 - N - i\omega^* k_p - (l_2^2 + k^{*2})k_p^2]^2 + 4l_2^2k_p^2(1 - i\omega^* k_p - k^{*2}k_p^2)}\right\}$$

$$k_3 = k_s, \quad l_2 = \sqrt{\alpha/\xi}, \quad \omega^* = \omega l_2^2 c_p/\alpha, \quad k^* = l_2 c_p\sqrt{\rho k/\alpha} \tag{2.3}$$

The quantities l_2, ω^* and k^* have the dimensions of length, while $N = (\beta/\xi)H$, $N \in (0,1)$ is a dimensionless parameter, the so-called "coupling number".[1–6] of the two branches of the square root we take the arithmetic value, i.e. $Re(\sqrt{z}) \geq 0$ for any complex z in a plane with a standard cut along the negative semiaxis.

The general solution of system (2.1) can now be written in the form

$$\begin{Vmatrix} U \\ V \\ \Phi \end{Vmatrix} = B_1Q_1 + B_2Q_2 + B_3Q_3 + B_4R_1 + B_5R_2 + B_6R_3 \tag{2.4}$$

$$Q_n = \begin{Vmatrix} \mathrm{ch}(\gamma_n y) \\ -\frac{\gamma_n}{is}\mathrm{sh}(\gamma_n y) \\ \frac{\gamma_n^2 + k_p^2 - s^2}{isH}\mathrm{ch}(\gamma_n y) \end{Vmatrix}, \quad R_n = \begin{Vmatrix} -\mathrm{sh}(\gamma_n y) \\ \frac{\gamma_n}{is}\mathrm{ch}(\gamma_n y) \\ -\frac{\gamma_n^2 + k_p^2 - s^2}{isH}\mathrm{sh}(\gamma_n y) \end{Vmatrix}, \quad n = 1, 2$$

$$Q_3 = \begin{Vmatrix} -\frac{\gamma_3}{is}\mathrm{ch}(\gamma_3 y) \\ -\mathrm{sh}(\gamma_3 y) \\ 0 \end{Vmatrix}, \quad R_3 = \begin{Vmatrix} \frac{\gamma_3}{is}\mathrm{sh}(\gamma_3 y) \\ \mathrm{ch}(\gamma_3 y) \\ 0 \end{Vmatrix}$$

The six unknown constants $B_1, B_2, \ldots, B_6$ are determined by using the six boundary conditions (2.2). Substituting expressions (2.4) into conditions (2.2) we obtain $B_4 = B_5 = B_6 = 0$. The remaining three constants are found from the following system of linear algebraic equations

$$\sum_{n=1}^{2} 2is\gamma_n \mathrm{sh}(\gamma_n h)B_n - (2s^2 - k_s^2)\mathrm{sh}(\gamma_3 h)B_3 = 0$$

$$\sum_{n=1}^{2} \gamma_n(\gamma_n^2 + k_p^2 - s^2)\mathrm{sh}(\gamma_n h)B_n = 0$$

$$\sum_{n=1}^{2} (k_p^2 - 2c^2s^2)\mathrm{ch}(\gamma_n h)B_n - 2isc^2\gamma_3\mathrm{ch}(\gamma_3 h)B_3 = \frac{isP(s)}{\lambda + 2\mu} \tag{2.5}$$

the solution of which has the form

$$B_n = \frac{\Delta_n(s)}{c^2\Delta(s)}, \quad n = 1, 2, 3; \quad \Delta(s) = (2s^2 - k_s^2)^2[\gamma_2(k_p^2 - k_2^2)\,\mathrm{sh}(\gamma_2 h)\,\mathrm{ch}(\gamma_1 h) -$$

$$-\gamma_1(k_p^2 - k_1^2)\,\mathrm{sh}(\gamma_1 h)\,\mathrm{ch}(\gamma_2 h)]\,\mathrm{sh}(\gamma_3 h) - 4s^2\gamma_3\,\mathrm{ch}(\gamma_3 h)\Delta_0(s)$$

$$\Delta_n(s) = \gamma_n(2s^2 - k_s^2)(k_p^2 - k_n^2)\,\mathrm{sh}(\gamma_n h)\,\mathrm{sh}(\gamma_3 h)\frac{isP(s)}{\lambda + 2\mu}, \quad n = 1, 2$$

$$\Delta_3(s) = 2is\Delta_0(s)\frac{isP(s)}{\lambda + 2\mu}, \quad \Delta_0(s) = \gamma_1\gamma_2(k_2^2 - k_1^2)\,\mathrm{sh}(\gamma_1 h)\,\mathrm{sh}(\gamma_2 h) \tag{2.6}$$

We can now determine the amplitude of the oscillations of the upper boundary ($y = h$). We obtain

$$V(s, h) = -\sum_{n=1}^{2}\frac{\gamma_n}{is}\,\mathrm{sh}(\gamma_n h)B_n - \mathrm{sh}(\gamma_3 h)B_3 = -\frac{P(s)}{\mu}L(s) \tag{2.7}$$

$$L(s) = -k_s^2\,\mathrm{sh}(\gamma_3 h)\frac{\Delta_0(s)}{\Delta(s)} \tag{2.8}$$

Turning to the last boundary condition of (1.4) using an inverse Fourier transformation and the convolution theorem, we obtain the fundamental integral equation

$$\int_{-a}^{a} p(\xi)K(x - \xi)d\xi = -\mu\upsilon_0, \quad |x| \le a; \quad K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} L(s)e^{-isx}ds, \tag{2.9}$$

the solution of which defines the contact pressure $p(x)$ in the contact region under the base of the punch $|x| < a$.

In the case when $N = \beta = 0$ we have $k_1^2 = k_p^2$, $k_2^2 = -1/l_2^2$, and the formula for the symbol of the kernel (2.8) reduces to the expression

$$L(s) = \frac{k_s^2\gamma_1\,\mathrm{sh}(\gamma_1 h)\,\mathrm{sh}(\gamma_3 h)}{(2s^2 - k_s^2)^2\,\mathrm{ch}(\gamma_1 h)\,\mathrm{sh}(\gamma_3 h) - 4s^2\gamma_1\gamma_3\,\mathrm{sh}(\gamma_1 h)\,\mathrm{ch}(\gamma_3 h)} \tag{2.10}$$

well known in the classical theory of elasticity.[9]

In the case of a very thick layer, i.e. when $h \to \infty$, expression (2.8) takes the form

$$L(s) = \frac{k_s^2\gamma_1\gamma_2(k_1^2 - k_2^2)}{(2s^2 - k_s^2)^2[\gamma_2(k_p^2 - k_2^2) - \gamma_1(k_p^2 - k_1^2)] - 4s^2\gamma_1\gamma_2\gamma_3(k_1^2 - k_2^2)} \tag{2.11}$$

and is identical with the expression obtained earlier for a porous half-plane.[10]

3. The properties of the kernel of the fundamental integral equation and a numerical method of solving it

The qualitative properties of the solution of Eq. (2.9) depend on the behaviour of the function $L(s)$ in the neighbourhood of the real axis $\mathrm{Im}(s) = 0$ in the plane of the, generally speaking, complex-valued variable s (see, for example, Refs 9, 11). In the problem in question $L(s)$ is a meromorphic function, which is a common property of problems for a layer of constant thickness.

We will estimate the behaviour of this function at zero. Obviously

$$L(0) = \frac{(k_1k_2/k_s^2)(k_1^2 - k_2^2)\,\mathrm{sh}(k_1 h)\,\mathrm{sh}(k_2 h)}{k_1(k_p^2 - k_1^2)\,\mathrm{sh}(k_1 h)\,\mathrm{ch}(k_2 h) - k_2(k_p^2 - k_2^2)\,\mathrm{sh}(k_2 h)\,\mathrm{ch}(k_1 h)} \tag{3.1}$$

The expression in the numerator vanishes only if $\mathrm{sh}(k_1h) = 0$ or $\mathrm{sh}(k_2h) = 0$, which is impossible, since the wave numbers k_1 and k_2 are complex and have a non-zero real part. In general, the denominator of expression (3.1) also does not vanish, since the vanishing of a complex-valued expression leads to a system of two equations (the real and imaginary parts are equal to zero) with respect to one real parameter h. Hence, in general the value of $L(0)$ is regular and non-zero, which is important when investigating the uniqueness of the solution of the integral equation considered.[9,11]

The asymptotic of the function $L(s)$ for large values of the argument is also easily estimated in explicit form:

$$L(s) = \frac{A_*}{s} + L_0(s), \quad A_* = \frac{k_s^2}{2(k_p^2 - k_s^2)}; \quad L_0(s) = O\left(\frac{1}{s^3}\right), \quad \mathrm{Re}(s) \to \pm\infty \tag{3.2}$$

where the parameter A_* is obviously finite and non-zero. The behaviour of the function $L(s)$ for intermediate values of the argument has been investigated numerically on the real axis. It turns out that the denominator of the meromorphic function considered does not vanish, and, consequently, the function $L(s)$ does not have singular points.

We will describe the property being investigated in more detail. The vanishing of the complex-valued denominator, i.e., the function $\Delta(s)$ means that its real and imaginary parts vanish simultaneously. For a numerical investigation, the change in the sign of this function when its argument increases with a fairly small constant step is usually the criterion that the real function vanishes. The vanishing of the complex-valued function should then indicate the existence of a point, on passing through which, the real and imaginary parts simultaneously change sign.

In the situation when a strict proof that the denominator is non-zero cannot be constructed, it merely remains to note that for the relations of the geometrical and physical parameters of the problem considered, as a result of a laborious numerical analysis it has not been possible to find a case when a change in the sign of both functions occurs.

Hence, at least in the cases considered, the function $L(s)$ does not have singularities in the neighbourhood of the real axis. In this case[9,11] the solution of the fundamental integral equation (2.9) is unique and can be constructed using well-known direct numerical methods.

The qualitative properties of the integral equations of dynamic contact problems are described in more detail in Ref. 9.

We will describe the main details of the numerical method used to solve the fundamental integral equation. We first represent the kernel of Eq. (2.9) in the form

$$
\begin{aligned}
K(x) &= \frac{1}{\pi}\int_0^\infty L(x)\cos(sx)ds = \frac{A_*}{\pi}\int_0^\infty \left[\frac{\cos(sx)-e^{-s}}{s}\right]ds + \\
&+\frac{1}{\pi}\int_0^\infty \left\{\left[L(s)-\frac{A_*}{s}\right]\cos(sx)ds + \frac{A_* e^{-s}}{s}\right\}ds = -\frac{A_*}{\pi}\ln|x| + K_0(x) \\
K_0(x) &= \frac{1}{\pi}\int_0^\infty \left\{\left[L(s)-\frac{A_*}{s}\right]\cos(sx)ds + \frac{A_* e^{-s}}{s}\right\}ds
\end{aligned}
\tag{3.3}
$$

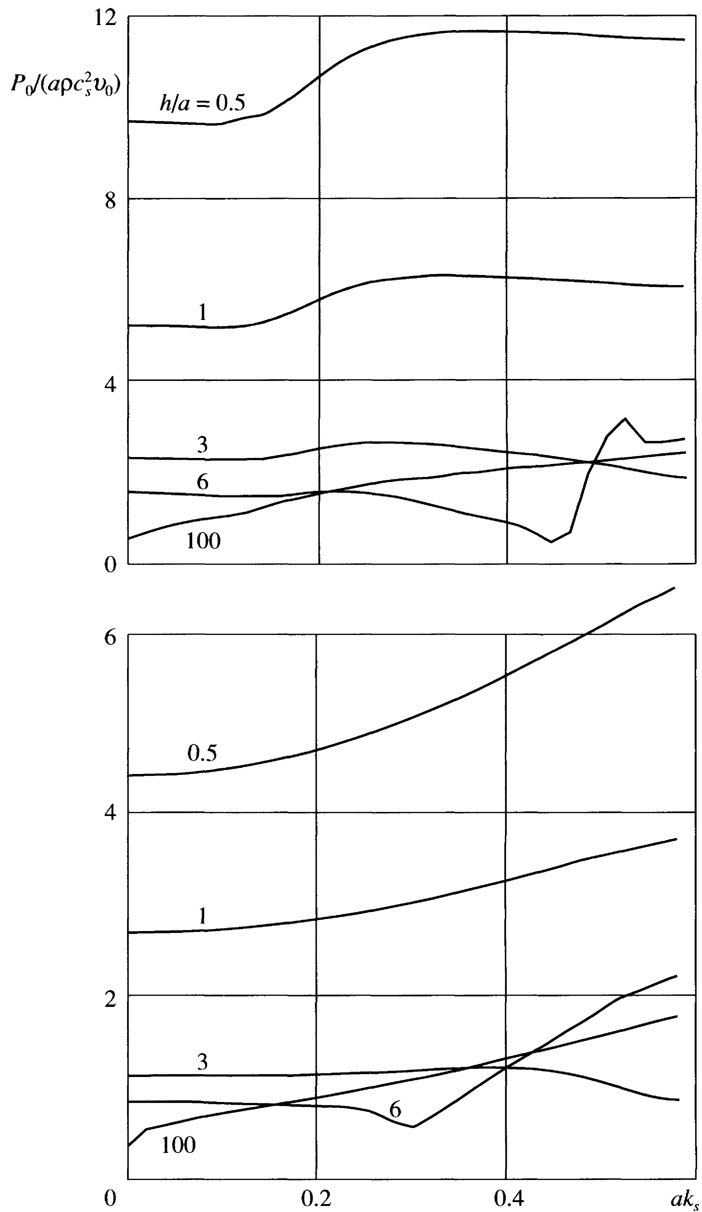
where the kernel $K_0(x)$ is a regular function (at least, differentiable) for all x.

The numerical approach used here for the numerical solution of Eq. (2.9) with kernel (3.3) is related to the methods employed to solve boundary integral equations.[12] It is based on a quadrature formula for subdividing the total interval of integration $(-a, a)$ into N small equal subintervals

$$
l_j = (-a+(j-1)h, -a+jh) = (x_j - h/2, x_j + h/2), \quad j = 1, 2, \ldots, N
$$

of length $h = 2a/N$, where $x_j = -a + (j - \frac{1}{2})h$. For the equation of the first kind considered, stable calculation is ensured provided that the weakly singular (logarithmic) part of the kernel is accurately integrable. This method is called the collocation method and, in more detail, is as follows. If the unknown function $p(x)$ is approximated by a constant quantity in each small subinterval l_j, the value of the left-hand side of Eq. (2.9) at the node x_i is equal to

$$
\begin{aligned}
&\int_{-a}^{a} g(\xi)K(x_i-\xi)d\xi = \sum_{j=1}^{n}\int_{x_j-h/2}^{x_j+h/2} g(\xi)K(x_i-\xi)d\xi \approx \sum_{j=1}^{n} g(x_j)\int_{x_j-h/2}^{x_j+h/2} K(x_i-\xi)d\xi = \\
&= \sum_{j=1}^{n} g(x_j)\int_{x_j-h/2}^{x_j+h/2}\left[-\frac{A_*}{\pi}\ln|x_i-\xi| + K_0(x_i-\xi)\right]d\xi \approx \\
&\approx \sum_{j=1}^{n} g(x_j)\left\langle -\frac{A_*}{\pi}\left\{\left(x_j+\frac{h}{2}-x_i\right)\left[\ln\left|x_j+\frac{h}{2}-x_i\right|-1\right]-\right.\right. \\
&\left.\left.-\left(x_j+\frac{h}{2}-x_i\right)\left[\ln\left|x_j+\frac{h}{2}-x_i\right|-1\right]\right\} + hK_0(x_i-x_j)\right\rangle =
\end{aligned}
\tag{3.4}
$$



$$= \sum_{j=1}^{n} a_{ij} g_j, \quad i = 1, 2, \ldots, N, \quad g_j = g(x_j)$$

Then, after equating the left and right-hand sides in Eq. (2.9) we arrive at the following linear algebraic system in $g_1, g_2, \ldots, g_N$

$$\sum_{j=1}^{n} a_{ij} g_j = f_i, \quad i = 1, 2, \ldots, N; \quad f_i = -\rho c_s^2 \upsilon_0 \tag{3.5}$$

In addition to the distribution of the contact stress, an important physical characteristic is also the relation between the applied load and the amplitude of the oscillations of the punch, defining the pliability of the porous foundation. The applied dynamic force is found from the equation

$$P_0 = \int_{-a}^{a} p(x) dx \tag{3.6}$$

In the figure we show the results of a calculation of the pliability as a function of the oscillation frequency for several characteristic values of the layered thickness in the case when $c^2 = c_s^2/c_p^2 = k_p^2/k_s^2 = 0.35$ for $l_2/a = 1$, $\omega^*/a = 5.0$, $k^*/a = 8.0$ and $N = 0.3$ (the upper part of the figure) and for $l_2/a = 0.1$, $\omega^*/a = 0.5$, $k^*/a = 0.8$ and $N = 0.55$ (the lower part). It is interesting to note that the pliability, as a rule, depends non-monotonically on the frequency. Only in the case of limiting thin layers is the frequency characteristic monotonically increasing. This can be explained by the fact that for extremely small thicknesses the foundation behaves as a Winkler foundation with fairly simple qualitative relations. An increase in the porosity (a higher value of N, the lower part of the figure) makes the frequency characteristics smoother. Moreover, the values of the dynamic pliability themselves are reduced in this case.

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